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Investigating measurement invariance by means of parameter instability tests for 2PL and 3PL models

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Abstract: M-fluctuations tests are a recently proposed method for detecting differential item functioning in Rasch models. This paper discusses a generalization of this method to two additional IRT models: The two-parametric logistic model and the three-parametric logistic model with a common guessing parameter. The Type I error rate and the power of this method were evaluated by a variety of simulation studies. The results suggest that the new method allows the detection of various forms of differential item functioning in these models, which also includes differential discrimination and differential guessing effects. It is also robust against moderate violations of several assumptions made in the item parameter estimation.

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Online Appendix A - The Derivation of Individual Scores for the 2PL and Constrained 3PL Models

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This document serves as the first online appendix for the study "Investigating Measurement Invariance by Means of Parameter Instability Tests for 2PL and 3PL models". Here, we will present the derivations of the score functions for the 2PL and constrained 3PL models. These functions were further used in the software implementation of the M-fluctuation tests for these models. In this text, the following notations are used: We assume a dataset of N persons responding to K items. The response of person j to item i is denoted by u_{ji} , with an observed value of 0 indicating an incorrect response and an observed value of 1 indicating a correct response. We further denote the probability of a correct response of person j to item i with $P(u_{ji})$, and the probability of an incorrect response with $Q(u_{ji})$. $\mathbf{a}, \mathbf{d}, \boldsymbol{\theta}$ are used as symbols for the vectors of the discrimination, intercept and person parameters, and \mathbf{u}_j denotes the vector of responses given by respondent j . Correspondingly, \mathbf{u}_i denotes the vector of responses given to item i .

In the context of marginal maximum likelihood estimation, the person parameters, which are denoted with θ_j , are assumed to be drawn from a specific distribution (in general, the normal distribution). This distribution is denoted by $g(\theta_j|\boldsymbol{\tau})$, with $\boldsymbol{\tau}$ being the parameters of the distribution g (e.g., the mean and variance of a normal distribution).

Using this notation defined in the previous section, the likelihood L of the observed data under the constrained 3PL model is given by (Baker & Kim, 2004, p. 160):

$$L = \prod_{j=1}^N P(\mathbf{u}_j)$$

This follows directly from the assumption of local independence made in this model.

Let x_i be an arbitrary item parameter (i.e., the slope parameter a_i , the intercept

parameter d_i or the pseudo-guessing-parameter c), the derivative of the log-likelihood with respect to x_i is directly obtained as (Baker & Kim, 2004, p. 160)

$$\frac{\partial \log L}{\partial x_i} = \frac{\partial \Psi(\mathbf{u}_i, \mathbf{a}, \mathbf{d}, c, \boldsymbol{\theta})}{\partial x_i} = \sum_{j=1}^N \frac{1}{P(\mathbf{u}_{j\cdot})} \frac{\partial}{\partial x_i} P(\mathbf{u}_{j\cdot}) \quad (\text{A1})$$

At this point, we will further consider that under the 2PL model, the probabilities $P(u_{ji})$ and $Q(u_{ji})$ are functions of the model parameters θ_j , a_i , d_i and c . In marginal maximum likelihood estimation, the θ_j parameter is further assumed to be taken from a specific distribution g in the population, e.g. the standard normal distribution. These assumptions lead to an alternative form for $P(\mathbf{u}_{j\cdot})$:

$$P(\mathbf{u}_{j\cdot}) = \int P(\mathbf{u}_{j\cdot} | \theta_j, a_i, d_i, c) g(\theta_j | \boldsymbol{\tau}) d\theta_j \quad (\text{A2})$$

Equation (A2) can now be inserted into Equation (A1). The resulting term can be simplified, as was shown by Baker and Kim (2004, p. 160-164); although these authors investigate the case that x_i is the discrimination parameter, their derivation generally holds. We finally obtain the following result for the log-likelihood:

$$\frac{\partial \Psi(\mathbf{u}_i, \mathbf{a}, \mathbf{d}, c, \boldsymbol{\theta})}{\partial x_i} = \sum_{j=1}^N \int (u_{ji} - P_{ji}) \cdot \frac{\partial P_{ji}}{\partial x_i} \cdot \frac{1}{P_{ji} \cdot Q_{ji}} \cdot P(\theta_j | \mathbf{u}_{j\cdot}, \mathbf{a}, \mathbf{d}, c, \boldsymbol{\tau}) d\theta_j$$

Here, we used P_{ji} as an abbreviation for $P(u_{ji})$ and Q_{ji} as an abbreviation for $Q(u_{ji})$.

At this point, we need the derivatives $\frac{\partial P_{ji}}{\partial x_i}$ for all item parameters in the constrained 3PL model. To present them, it is useful to introduce the following notation:

$$P_{ji}^* := \frac{\exp(a_i \cdot \theta_j + d_i)}{1 + \exp(a_i \cdot \theta_j + d_i)}$$

$$Q_{ji}^* := 1 - P_{ji}^*$$

The derivatives with respect to the a_i , d_i and c parameters are now given by:

$$\frac{\partial P_{ji}}{\partial a_i} = (1 - c) \cdot \theta_j \cdot \frac{\exp(a_i \theta_j + d_i)}{(1 + \exp(a_i \theta_j + d_i))^2} = (1 - c) \cdot \theta_j \cdot P_{ji}^* \cdot Q_{ji}^*$$

$$\frac{\partial P_{ji}}{\partial d_i} = (1 - c) \cdot \frac{\exp(a_i \theta_j + d_i)}{(1 + \exp(a_i \theta_j + d_i))^2} = (1 - c) \cdot P_{ji}^* \cdot Q_{ji}^*$$

$$\frac{\partial P_{ji}}{\partial c} = 1 - \frac{\exp(a_i \theta_j + d_i)}{(1 + \exp(a_i \theta_j + d_i))} = Q_{ji}^*$$

Finally, the derivatives of the log-likelihood are given by:

$$\frac{\partial \Psi(\mathbf{u}_{\cdot i}, \mathbf{a}, c, \mathbf{d}, \boldsymbol{\theta})}{\partial a_i} = \sum_{j=1}^N \int (u_{ji} - P_{ji}) \cdot (1 - c) \cdot \theta_j \cdot \frac{P_{ji}^* \cdot Q_{ji}^*}{P_{ji} \cdot Q_{ji}} \cdot P(\theta_j | \mathbf{u}_{j\cdot}, \mathbf{a}, c, \mathbf{d}, \boldsymbol{\tau}) d\theta_j$$

$$\frac{\partial \Psi(\mathbf{u}_{\cdot i}, \mathbf{a}, c, \mathbf{d}, \boldsymbol{\theta})}{\partial d_i} = \sum_{j=1}^N \int (u_{ji} - P_{ji}) \cdot (1 - c) \cdot \frac{P_{ji}^* \cdot Q_{ji}^*}{P_{ji} \cdot Q_{ji}} \cdot P(\theta_j | \mathbf{u}_{j\cdot}, \mathbf{a}, c, \mathbf{d}, \boldsymbol{\tau}) d\theta_j$$

$$\frac{\partial \Psi(\mathbf{u}_{\cdot i}, \mathbf{a}, c, \mathbf{d}, \boldsymbol{\theta})}{\partial c} = \sum_{j=1}^N \int (u_{ji} - P_{ji}) \cdot \frac{Q_{ji}^*}{P_{ji} \cdot Q_{ji}} \cdot P(\theta_j | \mathbf{u}_{j\cdot}, \mathbf{a}, c, \mathbf{d}, \boldsymbol{\tau}) d\theta_j$$

The score functions for the 2PL model are obtained by setting $c = 0$.

References

- Baker, F. B., & Kim, S.-H. (2004). *Item response theory: Parameter estimation techniques* (2nd ed.). Boca Raton, FL: CRC Press.